Exercises on linear programming—advanced

Doctoral course, 2026

(1) Show that any problem in inequality form can be put in standard form and vice-versa.

(2) *(Controlling the quality of solutions without resolving)* Consider the linear program

$$\begin{array}{rl} \text{minimize} & \boldsymbol{c}\cdot\boldsymbol{x} \\ \text{s.t.} & A'\boldsymbol{x} = \boldsymbol{b}' \\ & \displaystyle\sum_{i=1}^{n} x_i = 1 \\ & \boldsymbol{x} \in \mathbb{R}^n_+ \,. \end{array}$$

This problem is therefore in standard form

$$\begin{array}{rl} \text{minimize} & \boldsymbol{c}\cdot\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{R}^n_+ \,, \end{array}$$

with

$$A = \begin{pmatrix} & A' & \\ 1 & \cdots & 1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} \boldsymbol{b}' \\ 1 \end{pmatrix}.$$

Let B be a feasible basis of this problem and $\boldsymbol{r}$ the vector of reduced costs associated with this basis. Show that if $\boldsymbol{r} \ngeqslant \boldsymbol{0}$, then the optimal value v of this problem satisfies

$$\boldsymbol{c}_B^T A_B^{-1}\boldsymbol{b} + (\min_j r_j) \leqslant v \leqslant \boldsymbol{c}_B^T A_B^{-1}\boldsymbol{b}\,.$$

(3) *(An algorithmic proof of the strong duality theorem of linear optimization)* We know that there is a pivot rule that ensures that the simplex algorithm always ends.

* Deduce an alternative proof of the theorem of strong duality in linear optimization.

(4) *(How to get "non-degenerate" linear programs)* A basis B is *degenerate* if the corresponding solution has one or more entries equal to 0 among those indexed by B. A linear program is *degenerate* if it admits at least one degenerate feasible basis. Consider a linear program under the standard form:

$$\begin{array}{rl} \text{minimize} & \boldsymbol{c}\cdot\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{R}^n_+ \,, \end{array}$$

with $\boldsymbol{b} \in \mathbb{R}^m$.

* Show that there exist vectors $\boldsymbol{\beta}$ of arbitrarily small norm such that the linear program obtained by replacing $\boldsymbol{b}$ with $\boldsymbol{b}+\boldsymbol{\beta}$ is non-degenerate.

(5) *(Foundations of column generation)* Let (P) be the linear program

$$\begin{array}{rl} \text{minimize} & \boldsymbol{c}\cdot\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{R}^n_+ \,. \end{array}$$

Let $I \subseteq [n]$ and consider (P^I)

$$\begin{array}{rl} \text{minimize} & \boldsymbol{c}_I\cdot\boldsymbol{x}' \\ \text{s.t.} & A_I\boldsymbol{x}' = \boldsymbol{b} \\ & \boldsymbol{x}' \in \mathbb{R}^I_+ . \end{array}$$

Denote by (D) and (D^I) the dual problems of (P) and (P^I) respectively, and let $\tilde{\boldsymbol{y}}$ be an optimal solution of (D^I). Show that if $\tilde{\boldsymbol{y}}$ is a feasible solution for (D), then the optimal value of (P) coincides with that of (P^I).

This is the fundamental remark on which *column generation* relies, which is a useful method for solving linear programs with too many variables for being given directly to a solver.

(6) Consider two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ of $\mathbb{R}^n$ Show that there exist vectors $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}$, not necessarily distinct, such that

$$\begin{bmatrix} I_n & \boldsymbol{0} \\ \boldsymbol{x}^\top & 1 \end{bmatrix} \begin{bmatrix} I_n+\boldsymbol{u}\boldsymbol{v}^\top & -\boldsymbol{y} \\ \boldsymbol{0}^\top & 1 \end{bmatrix} \begin{bmatrix} I_n & \boldsymbol{0} \\ \boldsymbol{z}^\top & 1 \end{bmatrix} = \begin{bmatrix} I_n & \boldsymbol{0} \\ \boldsymbol{w}^\top & 1+\boldsymbol{v}^\top\boldsymbol{u} \end{bmatrix}.$$

Deduce from it the identity

$$\det(I_n + \boldsymbol{u}\boldsymbol{v}^\top) = 1 + \boldsymbol{v}^\top\boldsymbol{u}\,,$$

used in the proof of the expression of the Löwner–John ellipsoid.

(7) Consider a non-empty polytope $P = \{\boldsymbol{x} \in \mathbb{R}^n \colon A\boldsymbol{x} \geqslant \boldsymbol{b}\}$. Let t be any positive real number. Show that the polytope $P' = \{\boldsymbol{x} \in \mathbb{R}^n \colon A\boldsymbol{x} \geqslant \boldsymbol{b} - t\boldsymbol{1}\}$ has a non-empty interior.

(8) Let $G = (V, E)$ be an undirected graph, with costs c_e attached to the edges $e \in E$. A classical relaxation of the traveling salesman problem is given by the following linear program:

$$\begin{array}{rll} \text{minimize} & \displaystyle\sum_{e\in E} c_e x_e & \\ \text{s.t.} & \displaystyle\sum_{e\in\delta(v)} x_e = 2 & \forall v \in V \\ & \displaystyle\sum_{e\in\delta^-(S)} x_e \geqslant 2 & \forall S \in 2^V \setminus \{\varnothing, V\} \\ & 0 \leqslant x_e \leqslant 1 & \forall e \in E\,. \end{array}$$

Show that the optimal value of the linear program can be computed in polynomial time.

(9) Let $D = (V, A)$ be a directed graph, with a specified vertex $r \in V$ and positive costs c_a on the arcs $a \in A$. Consider the following linear program.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{a \in A} c_a x_a \\
\text{s.t.} \quad & \sum_{a \in \delta^-(S)} x_a \geqslant 1 \quad && \forall S \subseteq V \setminus \{r\} \\
& x_a \geqslant 0 && \forall a \in A\,.
\end{aligned}
$$

Show that the problem can be solved in polynomial time.

The motivation is the following. An *arborescence* is a directed graph whose underlying undirected graph is a tree and that has a vertex from which all other vertices can be reached by a directed path. It is known that the linear program models the problem of computing the minimum-cost spanning arborescence, which can thus be computed in polynomial time.