Exercises on linear programming—basics

Doctoral course, 2026

(1) Consider the linear program

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c} \cdot \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \geqslant \boldsymbol{b}\,. \end{aligned}$$

Suppose A and $\boldsymbol{b}$ have rational coefficients. Show that if the program has at least one optimal solution, then there exists an optimal solution with only rational coefficients.

(2) *(A refinery problem)* A refinery must supply two types A and B of gas per day from constituents 1, 2, and 3. We denote by $Q_{\max}$ the maximum quantity available daily:

constituent	$Q_{\max}$	unit cost
1	3000	3
2	2000	6
3	4000	4

gas	grade	unit selling price
A	$\leqslant$ 30% of 1	5.5
	$\geqslant$ 40% of 2	
	$\leqslant$ 50% of 3	
B	$\leqslant$ 50% of 1	4.5
	$\geqslant$ 10% of 2	

Formulate as a linear program the problem of determining the production plan that maximizes profit, knowing that all the production can be sold.

(3) For a vector $\boldsymbol{x}$, we denote by $x^{[i]}$ the ith largest number in the set $\{x_1, \ldots, x_n\}$. In particular, $x^{[1]}$ is the largest component of $\boldsymbol{x}$ and $x^{[n]}$ is the smallest. For $\boldsymbol{x} = (2, 1, 0, 1)$, we therefore have $x^{[2]} = x^{[3]} = 1$. Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{k} x^{[i]} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{R}^n_+\,, \end{aligned}$$

(a) Show that, for a fixed k, this problem can be formulated as a linear program by introducing one constraint for each subset of size k of $[n]$.

We can actually do better.

(b) Show that $x^{[1]} + x^{[2]} + \cdots + x^{[k]} = \inf_{t \in \mathbb{R}} kt + \sum_{i=1}^{n} \max(0, x_i - t)$.

(c) Deduce that we can actually write this problem as a linear program with at most $m+n$ constraints (not counting possible sign constraints), where m is the number of rows of A.

(4) Show that the problem of determining the largest ball in a polyhedron is a linear program.

(5) Let $A \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Show that there exists $\eta > 0$ such that for all $t \in (0, \eta)$ we have the following: "$P' = \{\boldsymbol{x} \in \mathbb{R}^n \colon A\boldsymbol{x} \geqslant \boldsymbol{b} - t\boldsymbol{1}\}$ has non-empty interior" is equivalent to "$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon A\boldsymbol{x} \geqslant \boldsymbol{b}\}$ non-empty." (Hint: use the Farkas lemma for the direct implication.)

(6) Let $\boldsymbol{a}^1, \boldsymbol{a}^2, \ldots, \boldsymbol{a}^m$ be m vectors of $\mathbb{R}^n$ and $b_1, b_2, \ldots, b_m$ be m real numbers. Throughout this exercise, we suppose given a vector $\boldsymbol{c} \in \mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$ such that $\boldsymbol{c} \cdot \boldsymbol{x} \geqslant \alpha$ for every $\boldsymbol{x}$ satisfying $\boldsymbol{a}^i \cdot \boldsymbol{x} \geqslant b_i$ for all $i \in [m]$. Such an inequality $\boldsymbol{c} \cdot \boldsymbol{x} \geqslant \alpha$ is a *valid inequality* for the system $\boldsymbol{a}^i \cdot \boldsymbol{x} \geqslant b_i\,, \forall i \in [m]$.

(a) Prove that there exists $\alpha' \geqslant \alpha$ such that $\boldsymbol{c} \cdot \boldsymbol{x} \geqslant \alpha'$ is simultaneously
- a valid inequality for the system $\boldsymbol{a}^i \cdot \boldsymbol{x} \geqslant b_i\,, \forall i \in [m]$.
- a nonnegative linear combination of the inequalities $\boldsymbol{a}^i \cdot \boldsymbol{x} \geqslant b_i$.

(Hint: use strong duality.)

The valid inequality $\boldsymbol{c} \cdot \boldsymbol{x} \geqslant \alpha$ is *tight* if it is satisfied as an equality for some $\boldsymbol{x}$ satisfying $\boldsymbol{a}^i \cdot \boldsymbol{x} \geqslant b_i$ for all $i \in [m]$.

(b) Explain why if the valid inequality is tight, α' can actually be chosen to be α.

(7) Consider the following "infinite" linear program (which plays a role in "robust optimization"):

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c} \cdot \boldsymbol{x} \\ \text{s.t.} \quad & A(\boldsymbol{\omega})\boldsymbol{x} \geqslant \boldsymbol{b} \quad \forall \boldsymbol{\omega} \in Q\,, \end{aligned}$$

where Q is a polytope and where the dependence of the matrix A to $\boldsymbol{\omega}$ is affine. Show that this program can be rewritten as an usual linear program, at the cost of adding one extra variable for each pair of a facet of Q and a row of A. (Hint: use strong duality.)

(8) A *doubly stochastic matrix* is a matrix with coefficients in $\mathbb{R}_+$ such that the sum of the coefficients on any row and any column is equal

to 1. Prove that every doubly stochastic matrix is the convex combination of permutation matrices.

(9) *(An algorithm for approximating optimal graph covers)* Given a graph $G = (V, E)$, a *cover* is a set C of vertices such that every edge $e \in E$ has at least one endpoint in C.

(a) Model the problem of determining a cover of minimum cardinality as a linear program where the variables are further constrained to be integer (this is called an *integer linear program*).

The version of this problem where the variables are no longer forced to be integer is the *continuous relaxation* of this problem.

(b) Propose a simple procedure to build a cover C, from an optimal solution of the continuous relaxation, with a size at most twice the optimal size of a cover.

This approach is relevant since the problem of determining a cover of minimum cardinality is NP-*hard*.

(10) We consider a bipartite graph G, whose edges form a set E. A matching is *perfect* if every vertex is incident to at least one edge in the matching. We assume given a weight function $w \colon E \to \mathbb{R}$.

(a) Model the problem of deciding whether there exists a perfect matching M and, if so, of computing one with minimal total weight $\sum_{e \in M} w(e)$ as a linear program with integer variables.

(b) Using a result from the course, show that we can safely discard the integrality constraints.

(11) An *interval matrix* is a matrix (a_{ij}) with coefficients in $\{0, 1\}$ such that for every j, if we have $a_{kj} = a_{\ell j} = 1$, then $a_{ij} = 1$ for every i between k and ℓ. Show that an interval matrix is always totally unimodular.

(12) A company has identified a set of possible activities. Activity j starts at time d_j, ends at time f_j, and brings profit b_j. At any given time, the maximum number of activities launched may not exceed c.

(a) Show that the problem of determining the activities to be undertaken to maximize total profit can be formulated as a linear optimization problem with integer variables.

(b) Using the result of exercise (11), show that we can safely discard the integrality constraints.

(13) Using the result of the exercise (11), show that in every finite collection $\mathcal{I}$ of closed intervals of $\mathbb{R}$, the maximum number of pairwise disjoint intervals is equal to the minimum number of points necessary to intersect every interval.

(14) *(The matrix rounding problem)* Consider a real matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ (with m rows and n columns). We denote by ℓ_i the sum $\sum_j a_{ij}$ of the terms in the ith row and c_j the sum $\sum_i a_{ij}$ of the terms in the jth column. We wish to round the a_{ij}, the ℓ_i, and the c_j so that simultaneously

- the sum of the roundings of the a_{ij} on each row i is equal to the rounding of ℓ_i
- the sum of the roundings of the a_{ij} on each column j is equal to the rounding of c_j.

When rounding a number a, you are free to choose whether it is "up" or "down," i.e., you can replace as you wish a by $\lceil a \rceil$ (smallest integer larger than or equal to a) or by $\lfloor a \rfloor$ (largest integer smaller than or equal to a), and this independently of the way the other numbers have been rounded. The objective of this problem is to show that it is always possible to round the matrix A in the desired way.

(a) With x_{ij} the rounding searched for the entry a_{ij}, give linear inequality constraints on

- x_{ij} based on $\lfloor a_{ij} \rfloor$ and $\lceil a_{ij} \rceil$ for all i, j.
- $\sum_j x_{ij}$ based on $\lfloor \ell_i \rfloor$ and $\lceil \ell_i \rceil$ for all i.
- $\sum_i x_{ij}$ based on $\lfloor c_j \rfloor$ and $\lceil c_j \rceil$ for all j.

(b) Deduce that $\boldsymbol{x} = (x_{ij})$ satisfies the linear inequalities

$$\boldsymbol{b} \leqslant \begin{pmatrix} I_{mn} \\ M \end{pmatrix} \boldsymbol{x} \leqslant \boldsymbol{u}$$

for some totally unimodular matrix $M \in \mathbb{R}^{(m+n)\times(mn)}$ and for some vectors $\boldsymbol{b}$ and $\boldsymbol{u}$ of $\mathbb{Z}^{mn+m+n}$ to be described. (We have denoted by I_{mn} the identity matrix with mn rows and mn columns.)

(c) Conclude that it is always possible to round the matrix A as desired.

(d) Show that we can even ensure that $|\sum_{ij} x_{ij} - \sum_{ij} a_{ij}| \leqslant 1$.